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General one-particle fluctuations of the ideal Bose gas

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Abstract. Fluctuations of general one-particle observables, such as density, momentum, angular momentum and position, etc, are studied and their distribution functions are explicitly obtained for all non-zero temperatures. The normalization exponent for the fluctuation operators is explicitly obtained for the three distinct phases of the ideal Bose gas: normal, critical and condensed.

1. Introduction

The history of the ideal Bose gas goes back to 1924, when Bose [1] obtained a microscopic derivation of the blackbody radiation law based on quantum statistical mechanics. Since then the ideal Bose gas has served as a simple, exactly solvable model for testing statistical mechanics as a theory of collective phenomena, such as the superfluid phase transition, and fundamental features such as the quantum equipartition law, the equivalence of ensembles, etc. Despite the simplicity of the model, deriving exact results for the ideal Bose gas is not always trivial. The first mathematically rigorous results were on the thermodynamic limit, see [2] and references therein. A detailed analysis of the model, including the explicit expression for the infinite-volume equilibrium states, the phase transition it undergoes, etc, was performed in [3].

In this paper we are interested in fluctuations of one-particle observables. The fluctuations are defined as the mathematical objects given by the infinite-volume limit of expressions of the form

$$F_\delta(A)_\Lambda = \frac{1}{|\Lambda|^{\frac{1}{2}+\delta}} \int_\Lambda \mathrm{d}x\, [A(x) - \langle A(x)\rangle] \tag{1.1}$$

where $A(x)$ is any local observable at position $x \in \mathbb{R}^3$, and $\langle\cdot\rangle$ is the thermal expectation value for the state in the thermodynamic limit and $|\Lambda|$ denotes the volume of the bounded domain $\Lambda \subset \mathbb{R}^3$. The infinite-volume limit in the above expression is taken in the sense of the non-commutative central limit theorem [4]. The problem is for every observable A to find the proper choice of the exponent δ so that the limit

$$\lim_{\Lambda\uparrow\mathbb{R}^3} \langle \exp \mathrm{i}\lambda F_\delta(A)_\Lambda\rangle \tag{1.2}$$

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exists for sufficiently small $\lambda \in \mathbb{R}$ and is non-degenerate. Then, if $\delta = 0$ one says that $A(x)$ has *normal fluctuations*; if $\delta \neq 0$ one says that $A(x)$ has *abnormal fluctuations*. Following the general theory [4], we define the fluctuation operator $F_\delta(A)$ by its characteristic function (1.2) and write

$$F_\delta(A) = \lim_{\Lambda\uparrow\mathbb{R}^3} F_\delta(A)_\Lambda. \tag{1.3}$$

It has been proved [4, 13] that the limits (1.3), for each A in some suitably chosen set, form an algebra of fluctuation operators given by a representation of the algebra of canonical commutation relations (CCR), or more generally, a Lie algebra of fluctuation operators.

In the case when A is a linear combination of Bose creation and annihilation operators $a^\#(x)$ of the free Bose gas the above programme has already been worked out in [5].

In this paper we concentrate on fluctuations of one-particle observables, such as the density $n(x) = a^+(x)a(x)$, the momentum $p(x) = -\frac{1}{2}\mathrm{i}[(\nabla a^+(x))a(x) - a^+(x)(\nabla a(x))]$, the angular momentum, etc (see section 2). In the literature one finds already a number of results on fluctuations of this type in the ideal Bose gas. First, in the work by Wreszinski [6] the total density fluctuations are computed in the normal phase. Ziff *et al* [7] computed in the canonical ensemble the fluctuations of the occupation number of the lowest energy level, in the special case when Λ is a rectangular box. They obtained $\delta = \frac{1}{6}$ for densities ρ larger than the critical density ρ_c of the Bose–Einstein condensation. This result was generalized to arbitrary domains by Buffet and Pulé [8]. Nachtergaele [9] computed the fluctuations of the total angular momentum for an interacting boson system at high temperatures, in which case $\delta = \frac{1}{3}$. The fluctuations of the one-particle angular momentum for the ideal Bose gas in a rotating bucket [10] were obtained in [11]. Furthermore, Fannes *et al* [12] studied the fluctuations of the total momentum for a system of interacting bosons in high-temperature, clustering equilibrium states and found that they are normal, i.e. $\delta = 0$. In the same work, the fluctuations of the total mean position (the centre of mass) were proved to be abnormal, with $\delta = \frac{1}{3}$. The limit fluctuation operators turned out to be commuting with each other and actually independent Gaussian random variables.

Here we consider the fluctuations of a general class of one-particle local observables, including all the particular cases treated in the above cited papers in the ideal Bose gas at arbitrary densities, $\rho < \rho_c$, $\rho = \rho_c$ and $\rho > \rho_c$. Explicitly, we consider the fluctuations of all self-adjoint operators which are the second quantized form of the operators $x^\tau p^\sigma$ on the one-particle space. Here x^τ is any monomial in the components of the position operator $x \in \mathbb{R}^3$,

$$x^\tau = \prod_{j=1}^{3}(x^j)^{\tau^j} \qquad \text{with } \tau \in \mathbb{N}^3 \qquad |\tau| = \sum_{j=1}^{3}\tau^j \geqslant 0 \tag{1.4}$$

and

$$p^\sigma = \prod_{j=1}^{3}\left(-\mathrm{i}\frac{\partial}{\partial x^j}\right)^{\sigma^j} \qquad \text{with } \sigma \in \mathbb{N}^3 \qquad |\sigma| = \sum_{j=1}^{3}\sigma^j \geqslant 0 \tag{1.5}$$

is an arbitrary monomial in the components of the momentum operator. For notational convenience, we put $\overline{h} = m = 1$; $\mathbb{N}$ is the set of natural numbers, including zero.

One of our main results is given by the following table of exponents δ:

If $\rho < \rho_c$ or $\sigma \neq 0$, then $\delta = \frac{1}{3}|\tau|$

If $\rho = \rho_c$ and $\sigma = 0$, then $\delta = \frac{1}{6} + \frac{1}{3}|\tau|$

If $\rho > \rho_c$ and $\sigma = 0$, then $\delta = \frac{1}{3} + \frac{1}{3}|\tau|$.

It is clear that the index δ increases with increasing density.

Note that the cases considered in [6, 9, 11, 12] are included in the above table. The results of [7] and [8] for the canonical ensemble do not fit within our scheme because they pertain to a different type of observables: the occupation number of the lowest energy level cannot be written as a one-particle operator of the type considered here, but is related to the field fluctuations (see [5]).

Moreover, we compute the distribution function (1.2) of the one-particle fluctuations. It turns out that all the fluctuation operators behave as an algebra of classical observables at non-zero temperatures. This can be seen from the above table of exponents δ in combination with the results of [13]. An example of such a classical behaviour of the fluctuations has also been obtained in [12] for the position and momentum in a clustering equilibrium state. If $\rho \neq \rho_c$ the distribution of the fluctuations is jointly Gaussian; if $\rho = \rho_c$ the distributions are not Gaussian if $\sigma = 0$. Finally, we mention that the limit $\Lambda \uparrow \mathbb{R}^3$ is taken by uniform dilation of a unit volume domain Λ_1 of arbitrary shape containing the origin. The obtained distributions depend explicitly on the shape of this domain.

2. The n-point truncated function

We consider an extremal translation invariant infinite-volume equilibrium grand canonical state ω of the free Bose gas at inverse temperature β. It is a quasi-free state over the algebra of canonical commutation relations (CCR), characterized by its one- and two-point truncated functions given by

$$\begin{aligned}
&\omega_T(a^+(x)) = c\mathrm{e}^{\mathrm{i}\theta} \qquad \omega_T(a(x)) = c\mathrm{e}^{-\mathrm{i}\theta} \\
&\omega_T(a^+(x), a^+(x')) = \omega_T(a(x), a(x')) = 0 \\
&\omega_T(a^+(x), a(x')) = r_z(x - x') \\
&\omega_T(a^\#(x_1), \ldots, a^\#(x_n)) = 0 \qquad \forall n \geqslant 3
\end{aligned}$$

where $a^\#(x)$ stands for either $a^+(x)$ or $a(x)$, and

$$r_z(x) = \sum_{n=1}^{\infty} \frac{z^n}{(2\pi\beta n)^{3/2}} \exp(-|x|^2/2n\beta). \tag{2.1}$$

The critical density ρ_c is given by $\rho_c = r_1(0) = (2\pi\beta)^{-3/2} g_{3/2}(1)$ where the function $g_{3/2}$ is given by $g_{3/2}(z) = \sum_{n=1}^{\infty} z^n/n^{3/2}$. The quantity c in equation (2.1) is defined in terms of the particle density ρ and critical density ρ_c by

$$c = \begin{cases} 0 & \text{if } \rho \leqslant \rho_c \\ \sqrt{\rho - \rho_c} & \text{if } \rho > \rho_c \end{cases}$$

so that c^2 is the density of the condensate. The activity z is the unique solution of the equation $\rho = (2\pi\beta)^{-3/2} g_{3/2}(z)$, if $\rho \leqslant \rho_c$, and $z = 1$, if $\rho \geqslant \rho_c$.

We consider the fluctuations of one-particle observables of a general type, the construction of which starts with the basic operators $x^\tau p^\sigma$ acting on the one-particle space $L^2(\mathbb{R}^3)$, where x^τ is the multiplication by the monomial (1.4) of degree $|\tau|$, $(Rx)^\tau = R^{|\tau|} x^\tau$ for any $R > 0$, and p^σ is defined in equation (1.5). With every such operator we associate a sequence of local approximations $a^R_{\sigma,\tau}$, which are self-adjoint operators on $L^2(\mathbb{R}^3)$:

$$a^R_{\sigma,\tau} = \tfrac{1}{2}(f_R x^\tau p^\sigma + p^\sigma f_R x^\tau). \tag{2.2}$$

Here f_R is the multiplication by a real $C_0^\infty(\mathbb{R}^3)$ function defined as follows. Let $\Lambda_1 \in \mathbb{R}^3$ be a bounded connected domain of unit volume, containing the origin of coordinates, and let for any $R > 0$, χ_R be the indicator function of its uniform dilation:

$$\Lambda_R = \{x \in \mathbb{R}^3 : x/R \in \Lambda_1\}. \tag{2.3}$$

Let f be any fixed $C_0^\infty(\mathbb{R}^3)$ function satisfying $f(x) \geqslant 0$ and $\int \mathrm{d}x f(x) = 1$, then

$$f_R(x) = \int_{\mathbb{R}^3} \mathrm{d}y\, \chi_R(y) f(x-y). \tag{2.4}$$

Considering these local approximations (2.2) is a matter of taking into account the influence of boundary conditions, which are situated in the type of functions f. The convergence of the limits $\mathbb{R} \to \infty$ will be independent of the particular choice of f.

We denote by $A^R_{\sigma,\tau}$ the second-quantized form of the $L^2(\mathbb{R}^3)$ operators $a^R_{\sigma,\tau}$. For instance, if one considers functions of one-particle positions only, then $\sigma = 0$ and $A^R_{0,\tau}$ is written formally as

$$A^R_{0,\tau} = \int_{\mathbb{R}^3} \mathrm{d}x\, f_R(x) x^\tau a^+(x) a(x). \tag{2.5}$$

If one considers observables linear in momentum operators, then $|\sigma| = \sum_{j=1}^3 \sigma^j = 1$, say $\sigma^i = \delta_{j,i}$, and

$$A^R_{j,\tau} = -\tfrac{1}{2}\mathrm{i} \int_{\mathbb{R}^3} \mathrm{d}x\, f_R(x) x^\tau [\partial_j a^+(x) a(x) - a^+(x)\partial_j a(x)]. \tag{2.6}$$

If $|\sigma| > 1$, then the corresponding derivatives should be applied to $a^+(x)$ and $a(x)$ according to a definite symmetrization rule, which is, however, of no relevance to the asymptotic behaviour ($R \to \infty$) of the expressions representing the fluctuations.

Let us now consider the truncated expectation ω_{T} (Ursell function) of n operators $A^R_{\sigma_k,\tau_k}$, $k = 1, \ldots, n$, in the extremal state ω. It has the general form

$$\omega_{\mathrm{T}}(A^R_{\sigma_1,\tau_1}, \cdots, A^R_{\sigma_n,\tau_n}) = \sum_{\gamma_1,\ldots,\gamma_{2n}} K_{\gamma_1,\ldots,\gamma_{2n}} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, I_{\gamma_1,\ldots,\gamma_{2n}}(x_1, \ldots, x_n) \prod_{k=1}^n f_R(x_k) x_k^{\tau_k}. \tag{2.7}$$

Here the sum runs over sets of $2n$ multi-indices $\{\gamma_1, \ldots, \gamma_{2n}\}$, such that $\gamma_{2k-1} + \gamma_{2k} = \sigma_k$, $k = 1, \ldots, n$, and

$$I_{\gamma_1,\ldots,\gamma_{2n}}(x_1, \ldots, x_n) = \partial^{\gamma_1}_{y_1} \ldots \partial^{\gamma_{2n}}_{y_{2n}} \omega_{\mathrm{T}}(a^+(y_1)a(y_2), \ldots, a^+(y_{2n-1})a(y_{2n}))\Big|_{\substack{y_1=y_2=x_1\\ \cdots \\ y_{2n-1}=y_{2n}=x_n}} \tag{2.8}$$

The sum over $\gamma_1, \ldots, \gamma_{2n}$, with suitably chosen constants $K_{\gamma_1,\ldots,\gamma_{2n}}$, performs the necessary symmetrization of the derivatives entering into the definition of the momentum operators, see equation (2.6).

Due to the quasi-free property of the state ω, the distributions

$$\omega_{\mathrm{T}}(a^+(x_1)a(x_1), \ldots, a^+(x_n)a(x_n)) \tag{2.9}$$

can be expressed as a sum of terms which are products of r_z- and δ-functions. The structure of the terms in this sum can be conveniently represented by subgraphs of the full graph $\mathcal{G}_n$ with vertices $x_1, \ldots, x_n$ and a distinguished vertex v. To every non-zero factor $\omega_{\mathrm{T}}(a^+(x_k)a(x_l))$, respectively $\omega_{\mathrm{T}}(a(x_k)a^+(x_l))$, we put into correspondence an oriented edge from x_k to x_l, respectively from x_l to x_k, if $k \neq l$, and a loop at x_k, if $k = l$; to every non-zero factor $\omega_{\mathrm{T}}(a^+(x_k)) = c\exp(+\mathrm{i}\theta)$, respectively $\omega_{\mathrm{T}}(a(x_k)) = c\exp(-\mathrm{i}\theta)$, we put into correspondence an oriented edge from x_k to v, respectively from v to x_k. We note that $c \neq 0$ implies $z = 1$. By the definition of the truncated expectation, it contains only terms which are represented by subgraphs of $\mathcal{G}_n$, such that their restriction to the set $x_1, \ldots, x_n$ is connected. Taking into account that there are exactly two edges incident with each vertex $x_k, k = 1, \ldots, n$, one concludes that the subgraphs with the above properties are oriented polygons which pass through all the vertices $x_1, \ldots, x_n$. To each polygon $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{\pi(1)}$, which does not pass through v, there corresponds, depending on the orientation, one of the following two terms in the expansion of (2.9):

$$\begin{aligned}&\omega_{\mathrm{T}}(a^+(x_{\pi(1)}), a(x_{\pi(2)})) \cdots \omega_{\mathrm{T}}(a^+(x_{\pi(n)}), a(x_{\pi(1)}))\\&\omega_{\mathrm{T}}(a(x_{\pi(1)}), a^+(x_{\pi(2)})) \cdots \omega_{\mathrm{T}}(a(x_{\pi(n)}), a^+(x_{\pi(1)}))\end{aligned} \tag{2.10}$$

where $\pi(\cdot)$ is a permutation of the indices $\{1, 2, \ldots, n\}$. If the vertex set of the polygon contains v, then the corresponding term in the expansion of (2.9) is given by one of the following two expressions,

$$\begin{aligned}&c^2\omega_{\mathrm{T}}(a^+(x_{\pi(1)}), a(x_{\pi(2)})) \cdots \omega_{\mathrm{T}}(a^+(x_{\pi(n-1)}), a(x_{\pi(n)}))\\&c^2\omega_{\mathrm{T}}(a(x_{\pi(1)}), a^+(x_{\pi(2)})) \cdots \omega_{\mathrm{T}}(a(x_{\pi(n-1)}), a^+(x_{\pi(n)}))\end{aligned} \tag{2.11}$$

depending on the orientation. Next, we remark that every factor in the expressions (2.10) and (2.11) contains one a^+ and one a, so that

$$\begin{aligned}&\omega_{\mathrm{T}}(a^+(x_k), a(x_l)) = r_z(x_k - x_l)\\&\omega_{\mathrm{T}}(a(x_k), a^+(x_l)) = r_z(x_l - x_k) + \delta(x_l - x_k).\end{aligned} \tag{2.12}$$

Therefore, every product of the form (2.10) or (2.11) generates a sum of terms in which the factors $\omega_{\mathrm{T}}(a(x_k), a^+(x_l))$ are replaced either by $r_z(x_l - x_k)$, or by $\delta(x_l - x_k)$. The sum of monomials, in which all the factors $\omega_{\mathrm{T}}(a(x_k), a^+(x_l))$ are replaced by $r_z(x_l - x_k)$ is referred to as the n-particle contribution to ω_{T}. More generally, a k-particle contribution to (2.10) or (2.11) is called a term with k factors $r_z(\cdot)$ and $n - k$ factors $\delta(\cdot)$.

Coming now to the structure of $I_{\gamma_1,\ldots,\gamma_{2n}}(x_1, \ldots, x_n)$, it becomes clear how the derivatives on the right-hand side of equation (2.8) act on terms of the form (2.10) and (2.11) with a given arrangement of a^+ and a. For example, the derivatives acting on $\omega_{\mathrm{T}}(a^+(y_{2k-1}), a(y_{2l}))$ give rise to a factor

$$\partial^{\gamma_{2k-1}}_{y_{2k-1}} \partial^{\gamma_{2l}}_{y_{2l}} \omega_{\mathrm{T}}(a^+(y_{2k-1}), a(y_{2l})) = (-1)^{|\gamma_{2l}|} \left(\partial^{\gamma_{2k-1}+\gamma_{2l}} r_z\right)(x_k - x_l). \tag{2.13}$$

In conclusion, the n-particle contributions to a term of type (2.10) have the general form

$$\pm \prod_{k=1}^{n} (\partial^{\nu_k} r_z)(x_{\pi(k)} - x_{\pi(k+1)}) \tag{2.14}$$

where $\pi(n + 1) = \pi(1)$, and ν_k equals either $\gamma_{2\pi(k)-1} + \gamma_{2\pi(k+1)}$ or $\gamma_{2\pi(k)} + \gamma_{2\pi(k+1)-1}$. As a consequence,

$$\sum_{k=1}^{n} \nu_k = \sum_{k=1}^{n} \sigma_k \qquad \text{and} \qquad 0 \leqslant |\nu_k| \leqslant |\sigma_{\pi(k)}| + |\sigma_{\pi(k+1)}|. \tag{2.15}$$

In the k-particle contributions with $k < n$ some of the factors $r_z(\cdot)$ in (2.14) are replaced by $\delta(\cdot)$. The n-particle contributions to a term of type (2.11) have the general form

$$\pm c^2 \prod_{k=1}^{n-1} (\partial^{\nu_k} r_z)(x_{\pi(k)} - x_{\pi(k+1)}) \tag{2.16}$$

where $\nu_1 = \sigma_{\pi(1)}$, $\nu_n = \sigma_{\pi(n)}$, and the remaining ν_k satisfy equation (2.15).

We consider explicitly the following two cases:

(i) fluctuations of monomials in the components of the one-particle position operator, see equation (2.5), for which the variance is

$$\omega_{\mathrm{T}}(A^R_{0,\tau}, A^R_{0,\tau}) = \rho \int \mathrm{d}x_1\, [f_R(x_1)x_1^\tau]^2 + \int \mathrm{d}x_1\, \mathrm{d}x_2\, f_R(x_1)x_1^\tau f_R(x_2)x_2^\tau \left[r_z^2(x_1-x_2) + 2c^2 r_z(x_1-x_2)\right] \tag{2.17}$$

(ii) fluctuations of the observable which is linear in the momentum operator (2.6), with variance given by

$$\begin{aligned}\omega_{\mathrm{T}}(A^R_{j,\tau}, A^R_{j,\tau}) = -\tfrac{1}{4} \int \mathrm{d}x_1\, \mathrm{d}x_2\, & f_R(x_1)x_1^\tau f_R(x_2)x_2^\tau \\ & \times\{-2[\partial_j r_z(x_1-x_2)]^2 + 2[r_z(x_1-x_2)+c^2]\partial_j^2 r_z(x_1-x_2) \\ & -2\partial_j r_z(x_1-x_2)\partial_j\delta(x_1-x_2) + \partial_j^2 r_z(x_1-x_2)\delta(x_1-x_2) \\ & +[r_z(x_1-x_2)+c^2]\partial_j^2\delta(x_1-x_2)\}.\end{aligned} \tag{2.18}$$

We do not explicitly consider the general case of $A^R_{\sigma,\tau}$ with $|\sigma| > 1$, because it does not yield anything new (see below) but a very heavy notational burden.

3. Fluctuations in the normal phase

The normal phase at $\rho < \rho_{\mathrm{c}}$ is characterized by the absence of condensate, i.e. $c = 0$ in expression (2.1), and by the exponential decay of the two-point truncated function $r_z(x)$ as $|x| \to \infty$. In this case we prove a rather general property.

Lemma 3.1. If $\rho < \rho_{\mathrm{c}}$, then for all $n > 2$

$$\lim_{R\to\infty} R^{-3n/2-\sum_k|\tau_k|}\omega_{\mathrm{T}}(A^R_{\sigma_1,\tau_1}, \ldots, A^R_{\sigma_n,\tau_n}) = 0. \tag{3.1}$$

Proof. From (2.7) it follows that the expression on the left-hand side of equation (3.1) is a finite sum of terms of the form

$$R^{-3n/2} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^n f_R(x_k)(R^{-1}x_k)^{\tau_k}\partial^{\nu_k} r_z(x_k - x_{k+1}) \tag{3.2}$$

where $x_{n+1} = x_1$ and $\sum \nu_k = \sum \sigma_k$, and we have analogous terms with some of the functions r_z replaced by δ-functions. Note that we have used the property $(R^{-1}x_k)^\tau = R^{-|\tau|}x_k^\tau$. We continue with the explicit treatment of expression (3.2), since the presence of δ-functions leads to trivial modifications of the argument. After the change of variables $(x_1, \ldots, x_n) \to (u_1, \ldots, u_{n-1}, y)$ with

$$u_k = x_k - x_{k+1} \qquad k = 1, \ldots, n-1 \qquad y = x_n/R \tag{3.3}$$

expression (3.2) becomes

$$R^{3-3n/2} \int \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1} \prod_{k=1}^{n-1} \partial^{\nu_k} r_z(u_k) \partial^{\nu_n} r_z\left(\sum_{k=1}^{n-1} u_k\right) \times \int \mathrm{d}y \prod_{k=1}^n f_R\left(R\left(y + \frac{1}{R}\sum_{l=k}^{n-1} u_l\right)\right)\left(y + \frac{1}{R}\sum_{l=k}^{n-1} u_l\right)^{\tau_k}. \tag{3.4}$$

Note that the function

$$u \equiv (u_1, \ldots, u_{n-1}) \to \prod_{k=1}^{n-1} \partial^{\nu_k} r_z(u_k) \partial^{\nu_n} r_z\left(\sum_{k=1}^{n-1} u_k\right) \tag{3.5}$$

is integrable and, therefore, we may apply the theorem of dominated convergence to the integral over u. For any fixed u it is clear that

$$\lim_{R\to\infty}\prod_{k=1}^{n} f_R\left(R\left(y+\frac{1}{R}\sum_{l=k}^{n-1}u_l\right)\right)\left(y+\frac{1}{R}\sum_{l=k}^{n-1}u_l\right)^{\tau_k} = \chi_{\Lambda_1}(y)\prod_{k=1}^{n} y^{\tau_k}. \qquad (3.6)$$

Therefore, the limit $R\to\infty$ of expression (3.4) equals zero for all $n>2$. □

Remark 3.1. The statement of lemma 3.1 is independent of the parameters $\{\sigma_1,\dots,\sigma_n\}$. As can be seen from the proof, this is a consequence of the fact that in the normal phase the integrability of the function (3.5) does not depend on the presence of the derivatives ∂^{ν_k}, $k=1,\dots,n$, with $\sum\nu_k=\sum\sigma_k$.

Remark 3.2. From the proof it is clear that lemma 3.1 also holds when the monomial x^τ is replaced by a homogeneous function $h_{|\tau|}(x)$ of degree $|\tau|$ which is bounded for any finite $|x|$. For example, it holds with $|\tau|=0$ for the local angle observable $\phi(x)=\arctan(x^{(2)}/x^{(1)})$.

Now, in order to prove that fluctuations do exist under the choice of the exponent $-\frac{3}{2}n-\sum_{k=1}^{n}|\tau_k|$ in lemma 3.1, we turn to the case $n=2$ and show that the limiting variance

$$\lim_{R\to\infty} R^{-3-2|\tau|}\omega_{\mathrm{T}}(A^R_{\sigma,\tau},A^R_{\sigma,\tau}) \qquad (3.7)$$

exists and is non-trivial. Since, as mentioned above, the choice of the parameters σ is irrelevant for the exponent in the normal phase, we confine ourselves to the explicit consideration of the two cases (2.5) and (2.6).

Lemma 3.2. If $\rho<\rho_{\mathrm{c}}$, then the following non-trivial limits exist:

(i) $\lim_{R\to\infty} R^{-3-2|\tau|}\omega_{\mathrm{T}}(A^R_{0,\tau},A^R_{0,\tau}) = \left[\int \mathrm{d}u\, r_z^2(u)+\rho\right]\int_{\Lambda_1}\mathrm{d}v\, v^{2\tau}$

(ii) $\lim_{R\to\infty} R^{-3-2|\tau|}\omega_{\mathrm{T}}(A^R_{j,\tau},A^R_{j,\tau})$

$$= \tfrac{1}{2}\left\{\int \mathrm{d}u\,[(\partial_j r_z(u))^2 - r_z(u)\partial_j^2 r_z(u)] - 2\partial_j^2 r_z(0)\right\}\int_{\Lambda_1}\mathrm{d}v\, v^{2\tau}.$$

Proof. First we consider the variance for a function of the position only, given by equation (2.17) at $c=0$:

$$\omega_{\mathrm{T}}(A^R_{0,\tau},A^R_{0,\tau}) = \rho\int \mathrm{d}x_1\,\left(f_R(x_1)x_1^\tau\right)^2 + \int \mathrm{d}x_1\,\mathrm{d}x_2\, f_R(x_1)x_1^\tau f_R(x_2)x_2^\tau r_z^2(x_1-x_2). \qquad (3.8)$$

Since the limit

$$\lim_{R\to\infty} f_R(Rx) = \chi_{\Lambda_1}(x) \qquad (3.9)$$

yields the indicator function of the unit-volume domain Λ_1, then

$$\lim_{R\to\infty} R^{-3-2\tau}\rho\int \mathrm{d}x\,[f_R(x)x^\tau]^2 = \lim_{R\to\infty} R^{-3}\rho\int \mathrm{d}x\,[f_R(x)(R^{-1}x)^\tau]^2 = \rho\int_{\Lambda_1}\mathrm{d}y\, y^{2\tau} \qquad (3.10)$$

is the one-particle contribution in equation (3.8).

In the analysis of the two-point contribution one can use the dominated convergence theorem and the fact that the function $x\to r_z^2(x)$ is integrable. Thus, by changing variables $u=x_1-x_2$, $v=x_2$ and rescaling $v\to Rv$, one obtains

$$\lim_{R\to\infty} R^{-3-2|\tau|}\int \mathrm{d}x_1\,\mathrm{d}x_2\, f_R(x_1)x_1^\tau f_R(x_2)x_2^\tau r_z^2(x_1-x_2)\int \mathrm{d}u\, r_z^2(u)\int_{\Lambda_1}\mathrm{d}v\, v^{2\tau}. \qquad (3.11)$$

Next, the variance of the observable which is linear in the momentum operator is given by equation (2.18) at $c = 0$:

$$\omega_{\rm T}(A^R_{j,\tau}, A^R_{j,\tau}) = \frac{1}{2}\int {\rm d}x_1\,{\rm d}x_2\, f_R(x_1)x_1^\tau f_R(x_2)x_2^\tau \times\{[\partial_j r_z(x_1-x_2)]^2 - r_z(x_1-x_2)\partial_j^2 r_z(x_1-x_2) + \partial_j r_z(x_1-x_2)\partial_j\delta(x_1-x_2) - \tfrac{1}{2}\partial_j^2 r_z(x_1-x_2)\delta(x_1-x_2) - \tfrac{1}{2} r_z(x_1-x_2)\partial_j^2\delta(x_1-x_2)\}. \qquad (3.12)$$

For the one-particle contribution, $\omega^{(1)}_{\rm T}(\cdot,\cdot)$, after change of variables and integration by parts, we obtain

$$\omega^{(1)}_{\rm T}(A^R_{j,\tau}, A^R_{j,\tau}) = \frac{1}{2}\int {\rm d}u\,{\rm d}v\, f_R(v+u/2)(v+u/2)^\tau f_R(v-u/2)(v-u/2)^\tau \times\left\{\partial_j r_z(u)\partial_j\delta(u) - \tfrac{1}{2}\partial_j^2 r_z(u)\delta(u) - \tfrac{1}{2} r_z(u)\partial_j^2\delta(u)\right\} = -\partial_j^2 r_z(0)\int {\rm d}v\,[f_R(v)v^\tau]^2 + \tfrac{1}{4} r_z(0)\int {\rm d}v\,\{\partial_j[f_R(v)v^\tau]\}^2. \qquad (3.13)$$

Hence,

$$\lim_{R\to\infty} R^{-3-2|\tau|}\omega^{(1)}_{\rm T}(A^R_{j,\tau}, A^R_{j,\tau}) = -\partial_j^2 r_z(0)\int_{\Lambda_1} {\rm d}v\, v^{2\tau}. \qquad (3.14)$$

The two-particle contribution $\omega^{(2)}_{\rm T}(\cdot,\cdot)$, after change of variables, takes the form

$$\omega^{(2)}_{\rm T}(A^R_{j,\tau}, A^R_{j,\tau}) = \frac{1}{2}\int {\rm d}u\,{\rm d}v\, f_R(v+u/2)(v+u/2)^\tau f_R(v-u/2)(v-u/2)^\tau \times\{[\partial_j r_z(u)]^2 - r_z(u)\partial_j^2 r_z(u)\}. \qquad (3.15)$$

Since the function in the braces is integrable, we rescale only the variable $v \to Rv$, as a result of which the right-hand side becomes

$$\frac{1}{2}R^{3+2|\tau|}\int {\rm d}u\,{\rm d}v\, f_R(R(v+u/2R))(v+u/2R)^\tau f_R(R(v-u/2R))(v-u/2R)^\tau \times\{[\partial_j r_z(u)]^2 - r_z(u)\partial_j^2 r_z(u)\}. \qquad (3.16)$$

Thus we obtain

$$\lim_{R\to\infty} R^{-3-2|\tau|}\omega^{(2)}_{\rm T}(A^R_{j,\tau}, A^R_{j,\tau}) = \frac{1}{2}\int {\rm d}u\,\{[\partial_j r_z(u)]^2 - r_z(u)\partial_j^2 r_z(u)\}\int_{\Lambda_1} {\rm d}v\, v^{2\tau}. \qquad (3.17)$$

Equations (3.14) and (3.17) complete the proof of the lemma. □

We can now formulate the result for the characteristic functions of the considered fluctuation operators.

Theorem 3.3. In the normal phase, $\rho < \rho_{\rm c}$, the characteristic function of the fluctuation operator

$$R^{-3/2-|\tau|}[A^R_{0,\tau} - \omega(A^R_{0,\tau})]$$

in the limit $R \to \infty$ is given by the expression

$$\lim_{R\to\infty} \omega\left(\exp\left\{{\rm i}\lambda R^{-3/2-|\tau|}\left[A^R_{0,\tau} - \omega(A^R_{0,\tau})\right]\right\}\right) = \exp\left\{-\frac{1}{2}\lambda^2\left[\int {\rm d}u\, r_z^2(u) + \rho\right]\int_{\Lambda_1}{\rm d}v\, v^{2\tau}\right\} \qquad (3.18)$$

and the characteristic function of the fluctuation operator

$$R^{-3/2-|\tau|}\left[A^R_{j,\tau} - \omega(A^R_{j,\tau})\right]$$

in the limit $R \to \infty$ is given by the expression

$$\lim_{R\to\infty} \omega\left(\exp\left\{\mathrm{i}\lambda R^{-3/2-|\tau|}\left[A^R_{j,\tau} - \omega(A^R_{j,\tau})\right]\right\}\right)$$
$$= \exp\left\{-\tfrac{1}{4}\lambda^2\left\{\int \mathrm{d}u\,[(\partial_j r_z(u))^2 - r_z(u)\partial_j^2 r_z(u)] - 2\partial_j^2 r_z(0)\right\}\int_{\Lambda_1}\mathrm{d}v\, v^{2\tau}\right\}. \tag{3.19}$$

Proof. This follows straightforwardly from lemmas 3.1 and 3.2, and the well known expansion of the characteristic function in terms of the truncated expectation values:

$$\omega(\mathrm{e}^{\mathrm{i}\lambda Q}) = \exp\sum_{n=1}^{\infty}\frac{(\mathrm{i}\lambda)^n}{n!}\omega_{\mathrm{T}}(\underbrace{Q;\ldots;Q}_{n\ \text{times}}). \qquad \square$$

Following the ideas of [4], we note that the above theorem proves the existence of the limits

$$F(A_{j,\tau}) = \lim_{R\to\infty} R^{-3/2-|\tau|}[A^R_{j,\tau} - \omega(A^R_{j,\tau})] \tag{3.20}$$

defined as non-commutative central limit operators acting on the GNS-space induced by the distributions in the right-hand side of equations (3.18) and (3.19), which are states on the algebra of fluctuation operators $F(A_{j,\tau})$.

It is interesting to remark that the algebra of fluctuation operators is Abelian in the case of $A_{j,\tau}$ with $\tau \in \mathbb{N}^3$. Indeed, for any such τ and τ' one obviously has

$$[A^R_{0,\tau}, A^R_{0,\tau'}] = 0 \tag{3.21}$$

already for finite R. Consider next the commutator $[A^R_{0,\tau}, A^R_{j,0}]$ which, up to boundary terms, is either zero, when $\tau^j = 0$, or has the form $A^R_{0,\tau'}$ with $\tau'^i = \tau^i$ for $i \in \{1,2,3\}\setminus\{j\}$ and $\tau'^j = \tau^j - 1$, when $\tau^j \geqslant 1$. Consider the non-trivial case, when $|\tau'| = |\tau| - 1$. Then we obtain the commutator

$$[R^{-3/2-|\tau|}(A^R_{0,\tau} - \omega(A^R_{0,\tau})), R^{-3/2}(A^R_{j,0} - \omega(A^R_{j,0}))] = R^{-1}(R^{-3-|\tau'|}A^R_{0,\tau'}).$$

On the basis of theorem 3.3, one takes the expectation value of the above equation in the state ω, and passing to the limit $R \to \infty$ obtains

$$[F(A_{0,\tau}), F(A_{j,0})] = \lim_{R\to\infty} R^{-1}\omega\left(R^{-3-|\tau'|}A^R_{0,\tau'}\right) = 0 \tag{3.22}$$

since

$$\lim_{R\to\infty}\omega\left(R^{-3-|\tau'|}A^R_{0,\tau'}\right)$$

is bounded.

Remark 3.3. By using the same argument one can prove the commutativity of the algebra of fluctuations in the critical and condensed phases, see sections 4 and 5. This means that no quantum effects are visible on the level of one-particle fluctuations. This also means that the characteristic function of a sum of two operators of the considered type equals the product of the corresponding characteristic functions. In particular, if the distributions of the fluctuations are Gaussian, as in the case of the normal phase, then they are jointly Gaussian.

In conclusion we mention that the Gaussian distributions (3.18) and (3.19) show an explicit shape dependence through the integral $\int_{\Lambda_1}\mathrm{d}v\, v^{2\tau}$. Of course, for translation invariant observables, i.e. for $\tau = 0$, no shape dependence appears, as expected in a clustering state (compare with the result for the critical regime). Space homogeneous quantities do not show any boundary conditions dependence if one has exponential decay of the correlation functions, as is the case in the normal phases.

4. Critical fluctuations

At the critical point $\rho = \rho_c(\beta)$ and the fugacity $z = 1$. In this case the function $r_1(x)$, see equation (2.1), can be written in the form

$$r_1(x) = \sum_{n=1}^{\infty} G(n\beta, x) \tag{4.1}$$

where

$$G(n\beta, x) = \frac{1}{(2\pi\beta n)^{3/2}} \exp\left(-\frac{|x|^2}{2\beta n}\right). \tag{4.2}$$

By comparing the above sum to the integral $\int_0^\infty \mathrm{d}t\, G(t\beta, x)$ and using the fact that the function $t \to G(t\beta, x)$ has only one maximum at $t_* = |x|^2/3\beta$, one obtains the following lower and upper bounds on $r_1(x)$:

$$\frac{1}{2\pi\beta|x|} - \left(\frac{3}{2\pi \mathrm{e}}\right)^{3/2} \frac{1}{|x|^3} \leqslant r_1(x) \leqslant \frac{1}{2\pi\beta|x|} + \left(\frac{3}{2\pi \mathrm{e}}\right)^{3/2} \frac{1}{|x|^3}. \tag{4.3}$$

Lemma 4.1. If $\rho = \rho_c, z = 1$, then

(i) for the position fluctuations, we have for all $n \geqslant 2$,

$$\lim_{R\to\infty} R^{-2n-n|\tau|}\omega_T(A^R_{0,\tau}, \ldots, A^R_{0,\tau}) = \frac{n!}{n(2\pi\beta)^n} \int_{\Lambda_1} \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{x_1^\tau \cdots x_n^\tau}{|x_1 - x_2| \cdots |x_{n-1} - x_n||x_n - x_1|}$$

(ii) for the momentum fluctuations, we have for all $n \geqslant 3$,

$$\lim_{R\to\infty} R^{-3n/2}\omega_T(A^R_{j,0}, \ldots, A^R_{j,0}) = 0$$

and for $n = 2$,

$$\lim_{R\to\infty} R^{-3}\omega_T(A^R_{j,0}, A^R_{j,0}) = \tfrac{1}{2}\left\{ \int \mathrm{d}u\, [(\partial_j r_1(u))^2 - r_1(u)\partial_j^2 r_1(u)] - 2\partial_j^2 r_1(0)\right\}.$$

Proof. Because $c = 0$, only the polygons which pass through the vertices $x_1, \ldots, x_n$ will contribute to right-hand side of the general expression (2.7) for the n-point truncated function. Since each of the $n!/n$ oriented polygons has an identical n-particle contribution, and since each replacement of the function $r_1(\cdot)$ by a δ-function reduces the number of integrations and, hence, the order of the term in $R \to \infty$, we obtain

$$\begin{aligned}\lim_{R\to\infty} R^{-2n-n|\tau|}\omega_T(A^R_{0,\tau}, \ldots, A^R_{0,\tau}) &= (n-1)! \lim_{R\to\infty} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^{n} f_R(Rx_k)x_k^\tau \\ &\quad \times Rr_1(R(x_1 - x_2)) \cdots Rr_1(R(x_{n-1} - x_n))Rr_1(R(x_n - x_1)) \\ &= \frac{(n-1)!}{(2\pi\beta)^n} \int_{\Lambda_1} \mathrm{d}x_1 \cdots \int_{\Lambda_1} \mathrm{d}x_n \frac{x_1^\tau \cdots x_n^\tau}{|x_1 - x_2| \cdots |x_{n-1} - x_n||x_n - x_1|}.\end{aligned} \tag{4.4}$$

In deriving the last equality we have used the pointwise convergence

$$\lim_{R\to\infty} Rr_1(Rx) = \frac{1}{2\pi\beta|x|} \qquad x \neq 0 \tag{4.5}$$

which follows from the bounds (4.3). This proves (i).

From the proof it is clear that lemma 4.1 also holds if the monomial x^τ is replaced by a homogeneous function $h_{|\tau|}(x)$ of degree $|\tau| \geqslant 0$ which is bounded for any finite $|x|$.

Now we turn to the proof of statement (ii). The variance, $n = 2$, is given by expression (3.12) at the critical fugacity $z = 1$. First we note that the one-particle contribution (3.13) can be treated along the same lines yielding equation (3.14) at $z = 1$. The limit $R \to \infty$ of the two-particle contribution (3.15) follows by the same argument as in the case of the normal phase, see lemma 3.2, since the functions $u \to [\partial_j r_1(u)]^2$ and $u \to r_1(u)\partial_j^2 r_1(u)$ are integrable. This extends the result (ii) of lemma 3.2 to the critical fugacity $z = 1$.

For $n \geqslant 3$, from equations (2.7)–(2.10) we obtain

$$\lim_{R\to\infty} R^{-3n/2}\omega_{\mathrm{T}}(A^R_{j,0}, \dots, A^R_{j,0}) = C_n \lim_{R\to\infty} R^{-3n/2} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^{n} [f_R(x_k)\partial_j^{\nu_k} r_1(x_k - x_{k+1})] \tag{4.6}$$

where C_n is a combinatorial coefficient, $x_{n+1} = x_1$, $0 \leqslant \nu_k \leqslant 2$ and $\sum_{k=1}^{n} \nu_k = n$. We have taken into account that the terms with some of the functions r_1 replaced by δ-functions give a lower-order contribution in $R \to \infty$. Let us define the sets,

$$S_m = \{(x_1, \dots, x_n) \in \mathbb{R}^{3n} \colon |x_m - x_{m+1}| = \max_{1\leqslant k\leqslant n} |x_k - x_{k+1}|\} \qquad m = 1, \dots, n \tag{4.7}$$

so that $\cup_{m=1}^{n} S_m = \mathbb{R}^{3n}$. Obviously,

$$\lim_{R\to\infty} R^{-3n/2} \left|\omega_{\mathrm{T}}(A^R_{j,0}, \dots, A^R_{j,0})\right| \leqslant nC_n \lim_{R\to\infty} R^{-3n/2} \max_{\nu_1,\dots,\nu_n}{}' \int_{S_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^{n} |f_R(x_k)\partial_j^{\nu_k} r_1(x_k - x_{k+1})| \tag{4.8}$$

where the primed maximum is taken under the above-mentioned constraints on $\nu_1, \dots, \nu_n$. Now we set $x_k - x_{k+1} = u_k$, $k = 1, \dots, n$, and in the integral over S_n make the change of variables $(x_1, \dots, x_n) \to (u_1, \dots, u_{n-1}, y)$, where

$$x_k = y + \sum_{l=k}^{n-1} u_k \qquad 1 \leqslant k \leqslant n-1 \qquad x_n = y. \tag{4.9}$$

In the new variables the domain of integration becomes

$$S_n = \left\{(u_1, \dots, u_{n-1}) \in \mathbb{R}^{3(n-1)} \colon |u_k| \leqslant \left|\sum_{l=1}^{n-1} u_l\right|, k = 1, \dots, n-1\right\} \times \mathbb{R}^3. \tag{4.10}$$

For any $\varepsilon > 0$, denote by $S^{>}_{n,\varepsilon}$ the subset of S_n in which $|u_1| > \varepsilon$, and by $S^{\leqslant}_{n,\varepsilon}$ the subset of S_n in which $|u_1| \leqslant \varepsilon$. Therefore, by choosing ε such that the functions $u \to |\partial_j^{\nu} r_1(u)|$, with $j = 1, 2, 3$ and $\nu = 1, 2$, are monotonically decreasing for all $|u| > \varepsilon$, in the integral over $S^{>}_{n,\varepsilon}$ one can use the inequality $(j = 1, 2, 3)$

$$\left|\partial_j^{\nu} r_1\left(-\sum_{k=1}^{n-1} u_k\right)\right| \leqslant |\partial_j^{\nu} r_1(u_1)| \qquad \nu = 0, 1, 2, \ |u_1| > \varepsilon. \tag{4.11}$$

Next, by extending the domain of integration $S^{>}_{n,\varepsilon}$, and taking into account that $|f_R(x)| \leqslant 1$ for all $x \in \mathbb{R}^3$, one obtains the upper bound

$$\lim_{R\to\infty} R^{-3n/2} \int_{S^{>}_{n,\varepsilon}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^{n} |f_R(x_k)\partial_j^{\nu_k} r_1(x_k - x_{k+1})| \leqslant \lim_{R\to\infty} R^{-3n/2} \int_{B(\varepsilon,qR)} \mathrm{d}u_1 \, |\partial_j^{\nu_1} r_1(u_1)| \left|\partial_j^{\nu_n} r_1\left(\frac{u_1}{n-1}\right)\right| \int_{\mathbb{R}}^{3} \mathrm{d}y \, |f_R(y)|$$

$$\times \prod_{k=2}^{n-1} \int_{B(qR)} \mathrm{d}u_k \, |\partial_j^{\nu_k} r_1(u_k)| \tag{4.12}$$

where

$$B(\varepsilon, R) = \{u \in \mathbb{R}^3 : \varepsilon \leqslant |u| \leqslant R\} \qquad B(R) \equiv B(0, R) \tag{4.13}$$

and q is any number larger than 2diamΛ_1. In the integral over $S_{n,\varepsilon}^{\leqslant}$ one can use the upper bound

$$|\partial_j^{\nu} r_1(u_1)| \leqslant \max_{\nu=0,1,2} \sup_{u\in\mathbb{R}^3} |\partial_j^{\nu} r_1(u)| \equiv M \tag{4.14}$$

and, again by extending the domain of integration, one obtains in the limit $R \to \infty$,

$$\lim_{R\to\infty} R^{-3n/2} \int_{S_{n,\varepsilon}^{\leqslant}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{k=1}^{n} |f_R(x_k) \partial_j^{\nu_k} r_1(x_k - x_{k+1})| \leqslant \varepsilon^3 (n-1)^3 |B(1)| M^2$$
$$\times \lim_{R\to\infty} R^{-3n/2} \int_{\mathbb{R}^3} \mathrm{d}y \, |f_R(y)| \prod_{k=2}^{n-1} \int_{B(qR)} \mathrm{d}u_k |\partial_j^{\nu_k} r_1(u_k)|. \tag{4.15}$$

Obviously, if the right-hand side of equation (4.12) vanishes, then (4.15) vanishes too. Therefore, it suffices to consider the limit in (4.12). For the integral over y one obtains after rescaling $y \to Ry$

$$\lim_{R\to\infty} R^{-3} \int_{\mathbb{R}^3} \mathrm{d}y \, |f_R(y)| = |\Lambda_1| = 1. \tag{4.16}$$

In the integrals over u_k, $k = 1, \ldots, n-1$ we take into account the existence of the limits

$$\lim_{R\to\infty} R^2 \partial_j r_1(Ru) = -\frac{u^j}{2\pi\beta|u|^3} \qquad \lim_{R\to\infty} R^3 \partial_j^2 r_1(Ru) = \frac{3(u^j)^2 - |u|^2}{2\pi\beta|u|^5} \qquad \forall u \neq 0. \tag{4.17}$$

Therefore, as $R \to \infty$,

$$\int_{B(\varepsilon,qR)} \mathrm{d}u_1 \, |\partial_j^{\nu_1} r_1(u_1)| \left| \partial_j^{\nu_n} r_1 \left(\frac{u_1}{n-1} \right) \right| = \begin{cases} \mathrm{O}(R) & \nu_1 + \nu_n = 0 \\ \mathrm{O}(\log R) & \nu_1 + \nu_n = 1 \\ \mathrm{O}(1) & \nu_1 + \nu_n \geqslant 2. \end{cases} \tag{4.18}$$

Similarly, for $k = 2, \ldots, n-1$,

$$\int_{B(qR)} \mathrm{d}u_k \, |\partial_j^{\nu_k} r_1(u_k)| \leqslant M\varepsilon^3 |B(1)| + \int_{B(\varepsilon,qR)} \mathrm{d}u \, |\partial_j^{\nu_k} r_1(u)| = \begin{cases} \mathrm{O}(R^{2-\nu_k}) & \nu_k = 0, 1 \\ \mathrm{O}(\log R) & \nu_k = 2. \end{cases} \tag{4.19}$$

From (4.16), (4.18) and (4.19) it follows that the right-hand side of equation (4.12) attains its highest order in R, namely $\mathrm{O}(R^{1-n/2})$, when $\nu_k = 1$, $k = 1, \ldots, n$, which yields the proof of statement (ii) and the lemma. □

Theorem 4.1. The position fluctuations at criticality are non-Gaussian; the characteristic function of the fluctuation operator

$$R^{-2-|\tau|} [A_{0,\tau}^R - \omega(A_{0,\tau}^R)]$$

in the limit $R \to \infty$ is given by the expression

$$\lim_{R\to\infty} \omega(\exp\{\mathrm{i}\lambda R^{-2-|\tau|} [A_{0,\tau}^R - \omega(A_{0,\tau}^R)]\}) = 1 + \mathrm{tr}\left\{ -\frac{\mathrm{i}\lambda K_\tau}{2\pi\beta} - \log\left(1 - \frac{\mathrm{i}\lambda K_\tau}{2\pi\beta}\right) \right\} \tag{4.20}$$

where K_τ is the operator defined by

$$(K_\tau\psi)(x) := \int_{\Lambda_1} \mathrm{d}y \frac{y^\tau}{|x-y|}\psi(y) \tag{4.21}$$

for all $\psi \in L^2(\mathbb{R}^3)$.

Proof. The proof follows from equation (4.4), the fact that K_τ^2 is trace-class and the upper bound

$$|\operatorname{tr} K_\tau^n| \leqslant (\operatorname{tr} K_\tau^2)\|K_\tau\|^{n-2} \qquad \forall n \geqslant 2 \tag{4.22}$$

where

$$\|K_\tau\| \leqslant \left(\sup_x \int_{\Lambda_1} \mathrm{d}y \, \frac{y^{2\tau}}{|x-y|^2}\right)^{1/2} < \infty \tag{4.23}$$

and $\|K_\tau\|$ is the L^2-norm of K_τ. □

Remark 4.1. Theorem 4.1 holds also for fluctuation operators in which the monomial x^τ is replaced by a homogeneous function $h_{|\tau|}(x)$ of degree $|\tau| \geqslant 0$ which is bounded for any finite $|x|$. With the corresponding change in the definition (4.21) of the operator K_τ, the above result includes, as particular cases, the critical fluctuations of the density and the angle observables. The non-Gaussian distribution of these fluctuations and the exponent $\delta = \frac{1}{6} + |\tau|/3$ have not been computed before, not even for the density $|\tau| = 0$. Note also that the dependence on the shape of the unit volume Λ_1 is much more inherent in the distribution for the critical phase, than for the normal one.

5. Fluctuations in the condensed phase

For any $\beta > 0$ and densities larger than the critical one, $\rho > \rho_c(\beta)$, one has $z = 1$ and $\rho = c^2 + \rho_c$, where $c^2 > 0$ is the condensate density, $\rho_c = r_1(0)$, with $r_1(x)$ given by expressions (4.1)–(4.2). Note that the properties of the function $x \to r_1(x)$, which have been used in section 4, are qualitatively the same for all finite $\beta > 0$.

Lemma 5.1. If $\rho > \rho_c, z = 1$, then

(i) for the position fluctuations, we have for all $n \geqslant 2$,

$$\lim_{R\to\infty} R^{-5n/2-n|\tau|}\omega_T(A_{0,\tau}^R, \ldots, A_{0,\tau}^R) = 0$$

and for $n = 2$,

$$\lim_{R\to\infty} R^{-5-2|\tau|}\omega_T(A_{0,\tau}^R, A_{0,\tau}^R) = \frac{c^2}{\pi\beta}\int_{\Lambda_1}\mathrm{d}x_1\int_{\Lambda_1}\mathrm{d}x_2\frac{x_1^\tau x_2^\tau}{|x_1-x_2|}$$

(ii) for the momentum fluctuations, we have for all $n \geqslant 3$,

$$\lim_{R\to\infty} R^{-3n/2}\omega_T(A_{j,0}^R, \ldots, A_{j,0}^R) = 0$$

and for $n = 2$,

$$\lim_{R\to\infty} R^{-3}\omega_T(A_{j,0}^R, A_{j,0}^R) = \frac{1}{2}\left\{\int \mathrm{d}u\,[(\partial_j r_1(u))^2 - r_1(u)\partial_j^2 r_1(u)] - 2\partial_j^2 r_1(0)\right\}$$
$$+\frac{c^2}{(2\pi)^3\beta}\int \mathrm{d}k\,\left(\frac{k^j}{|k|}\right)^2\left|\int_{\Lambda_1}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}\right|^2.$$

Proof. Note that the n-point truncated function for the position fluctuations in the condensed phase is divided by a higher power of R, namely $R^{5n/2+n|\tau|}$, compared to $R^{2n+n|\tau|}$ for the critical fluctuations considered in lemma 4.1. Therefore, all the contributions which are not proportional to the condensate density c^2 will vanish in the limit $R \to \infty$. It remains to consider the n-particle contribution from the $n!$ oriented polygons which pass through all the vertices $x_1, \ldots, x_n$ and v:

$$\lim_{R\to\infty} R^{-5n/2-n|\tau|}\omega_{\mathrm{T}}(A^R_{0,\tau}, \ldots, A^R_{0,\tau}) = n!c^2 \lim_{R\to\infty} R^{-n/2+1} \int \mathrm{d}x_1 \cdots dx_n \prod_{k=1}^{n} f_R(Rx_k)x_k^\tau \\ \times Rr_1(R(x_1 - x_2)) \cdots Rr_1(R(x_{n-1} - x_n)) = 0 \tag{5.1}$$

for all $n \geqslant 3$, due to the limit (4.5). Since the k-particle contributions with $k < n$ are of lower order in R, they vanish too. The expression for the variance follows by the above argument for $n = 2$. This proves statement (i).

The proof of statement (ii), in the part concerning $n \geqslant 3$, goes along the same lines as for the position fluctuations, by using the estimates of equation (4.19). Finally, for computing the variance of the momentum fluctuations, we turn back to the general expression (2.18). In addition to the terms evaluated in lemma 4.1 (ii), we have to consider

$$\lim_{R\to\infty} R^{-3}\omega_{\mathrm{T}}(A^R_{j,0}, A^R_{j,0}) = -\tfrac{1}{2}c^2 \lim_{R\to\infty} R^{-3} \int \mathrm{d}x_1 \int \mathrm{d}x_2\, f_R(x_1) f_R(x_2) \\ \times \left[\partial_j^2 r_1(x_1 - x_2) + \tfrac{1}{2}\partial_j^2 \delta(x_1 - x_2)\right]. \tag{5.2}$$

The one-particle term, containing the δ-function, is readily shown to give no contribution. By using Fourier transforms and rescaling the integration variable, the two-particle term in equation (5.2) can be written as

$$\frac{c^2}{2(2\pi)^3} \lim_{R\to\infty} \int \mathrm{d}k \left| R^{-3} \hat{f}_R\left(\frac{k}{R}\right)\right|^2 \varphi_j\left(\frac{k}{R}\right) \tag{5.3}$$

where $\hat{f}_R(k)$ is the Fourier transform of $f_R(x)$ and

$$\varphi_j(k) = \frac{(k^j)^2}{\exp(\beta k^2/2) - 1}. \tag{5.4}$$

Clearly, φ_j is a positive bounded function, $0 \leqslant \varphi_j(k) \leqslant 2/\beta$, and pointwise

$$\lim_{R\to\infty} \varphi_j\left(\frac{k}{R}\right) = \frac{2}{\beta}\left(\frac{k^j}{|k|}\right)^2. \tag{5.5}$$

On the other hand,

$$\lim_{R\to\infty} R^{-3}\hat{f}_R\left(\frac{k}{R}\right) = \lim_{R\to\infty} \int \mathrm{d}x\, f_R(Rx)\mathrm{e}^{\mathrm{i}kx} = \int_{\Lambda_1} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} \in L^2(\mathbb{R}^3). \tag{5.6}$$

Hence one easily derives (ii) for $n = 2$. □

Finally, we have

Theorem 5.2. In the condensed phase, $\rho > \rho_{\mathrm{c}}$, the fluctuations of the position and momentum observables are Gaussian, with variances given by lemma 5.1.

Remark 5.1. All the fluctuations of the studied type are Gaussian in the condensed phase. This might look to be in contradiction with the results of [8] concerning the fluctuations of the density of particles at the lowest energy level. However, the latter are fluctuations of the square of the Bose field, rather than of one-particle observables. Therefore, the corresponding distributions can be totally different.

Remark 5.2. From remark 3.3 it is clear that in the condensed phase there are also no quantum effects visible at the level of one-particle fluctuations, in the sense that all fluctuation operators of the considered type commute pairwise.

6. Discussion

Our study of a general type of one-particle fluctuations in the ideal Bose gas complements the study of the field fluctuations given in [5]. We have obtained the volume scale, given by the normalization exponent δ, at which these fluctuations appear in the three distinct phases of the free Bose gas: normal, critical and condensed. Our results include as particular cases the fluctuations of the particle density, of position, angle, momentum and angular momentum. We have shown that momentum fluctuations are always normal Gaussian, even at the critical point. One of the remarkable results is the non-Gaussian distribution explicitly obtained for the critical fluctuations of the position type observables, including the particle density. Furthermore, it turns out that all the limit fluctuation operators commute pairwise and, therefore, no quantum effects are visible at the level of one-particle fluctuations, except in the individual distributions.

Although the validity of the explicit results obtained here is limited to the ideal Bose gas, the arguments used in the important lemmas 3.1 and 4.1 are based on rather general features, such as the integrability of the two-point function and the behaviour of the truncated n-point functions at infinity. These properties can be used in a study of the fluctuations in interacting Bose systems and, in particular, in various models of the general phenomenon of Bose–Einstein condensation (see e.g. [12]).

There still remain some open problems, even in the case of the free Bose gas. First of all, there is the problem of the fluctuations in the ground state. One way of approaching the ground state is by taking the limit of temperatures T tending to zero. A quick inspection reveals that the resulting fluctuations may depend on the particular way in which T tends to zero, for example, by setting $T \sim V^{-\alpha}$, with some $\alpha > 0$. It is not clear whether the normalization exponent for the fluctuations δ is independent of the parameter α. Another open question is whether the algebra of fluctuation operators will remain Abelian in the limit $T \to 0$. Of course, the fluctuations of two- and higher-particle observables still remain to be studied. According to physical intuition, it is to be expected that higher-particle fluctuations behave more normally than lower-particle ones. Can one verify this on models, or can one find a mathematically rigorous argument for its general validity?

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